From tree matching to sparse graph alignment.

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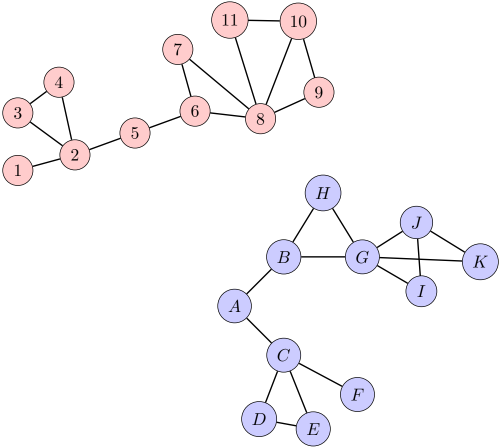
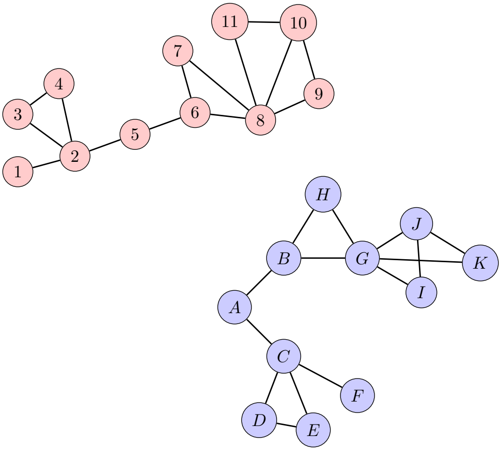
ÉCOLE NORMALE SUPÉRIEURE
PSL RESEARCH UNIVERSITY PARIS

Introduction: the graph isomorphism problem



Introduction: the graph isomorphism problem

Question: Given two graphs $G = (V, E)$ and $G' = (V', E')$, is there a *graph isomorphism*, i.e. a bijection $f : V \to V'$ such that $(i, j) \in E \iff (f(i), f(j)) \in E'$?



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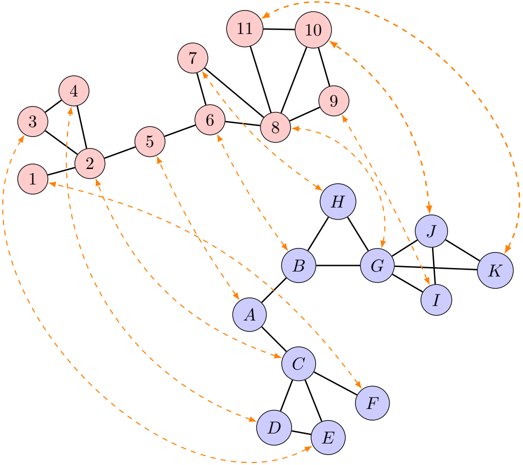
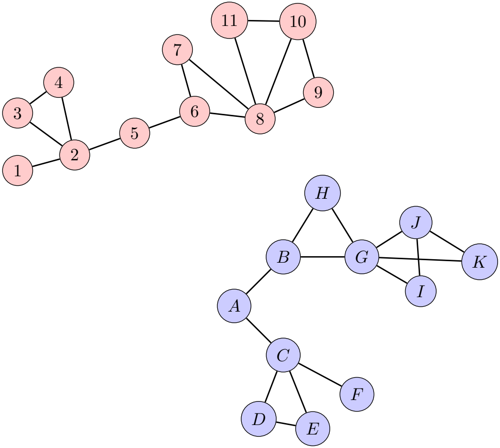


Problem in NP, thought to be neither in P nor NP-complete.

Introduction: graph alignment



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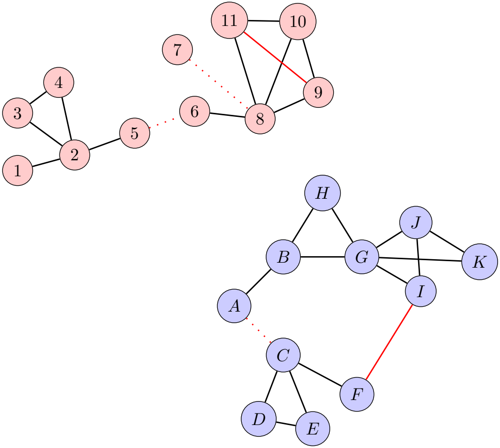
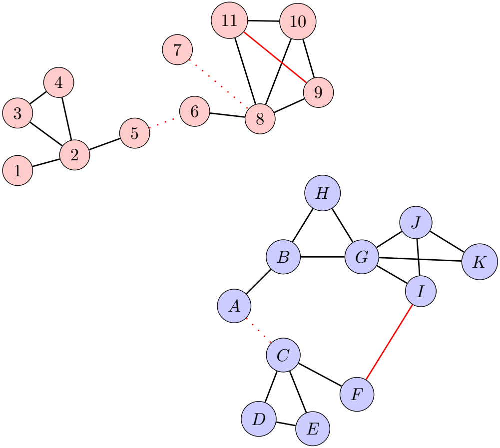


Introduction: graph alignment

Relaxed version: Is there a bijection $f : V \to V'$ that *preserves most edges*?



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Formally: f minimizes

$$\sum_{i=1}^{n} \left(\mathbf{1}_{(i,j)\in E} - \mathbf{1}_{(f(i),f(j))\in E'} \right) .$$

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$\longrightarrow$ instance of the NP-hard quadratic assigment problem (QAP):

$$\max_{\Pi} \mathrm{Tr} \left(G \Pi G' \Pi^\top \right),$$

where Π runs over all permutation matrices.

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- Shuffle labels of G_2' uniformly at random to form G_2. Formally, $G_2 = \Pi^\top G_2' \Pi$, where $\Pi = \Pi_\sigma$ is the matrix of a uniform permutation σ.

Informational and computational thresholds

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Exact recovery of σ:

- Information-theoretically feasible iff $nps = \log n + \omega(1)$ [Cullina-Kiyavash'16].
- Polynomial time feasible if $np \geq (\log n)^{\alpha}$ and $1 - s \leq (\log n)^{-\beta}$[Ding et al.'18].
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This work: partial recovery of σ

- Polynomial-time recovery, in sparse regime $np = O(1)$.
- Relaxed objective: find a one-to-one $\hat{\sigma}$ from G_1, G_2, such that

$$\mathrm{overlap}(\hat{\sigma}) := \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\hat{\sigma}(i)=\sigma(i)} = \Omega(1),$$

  and

$$\frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\hat{\sigma}(i)\neq\sigma(i)} = o(1).$$

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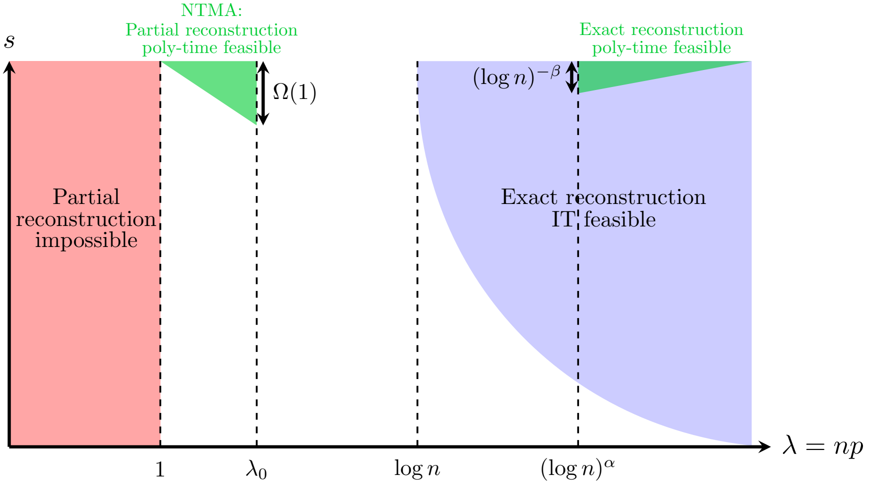

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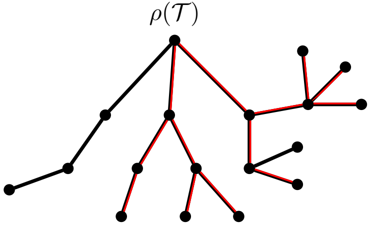

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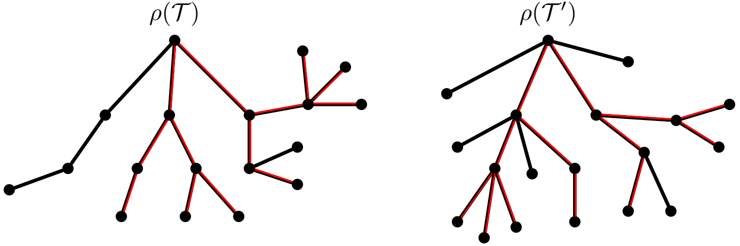

Recursive computation:

$$\mathcal{W}_d(\mathcal{T}, \mathcal{T}') = \sup_{\mathfrak{m}} \sum_{(i,u)\in\mathfrak{m}} \mathcal{W}_{d-1}(\mathcal{T}_{i\leftarrow\rho(\mathcal{T})}, \mathcal{T}'_{u\leftarrow\rho(\mathcal{T}')}).$$

Intuition for main result

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Theorem

For $\lambda \in (1, \lambda_0]$ and $s \in (s^*(\lambda), 1]$, then there exists $\gamma < \lambda s$ such that

$$\mathcal{W}_d(\mathcal{T}, \mathcal{T}') \ll \gamma^d \text{ as } d \to \infty.$$

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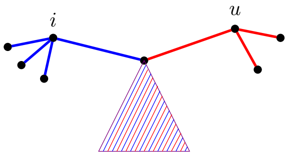

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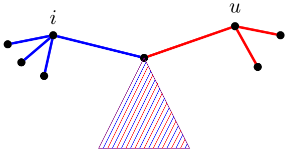
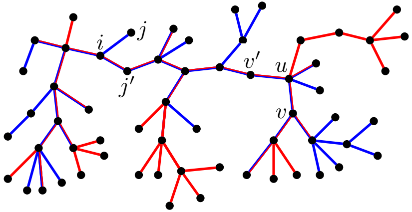


'Dangling trees trick': look at both $\mathcal{W}_{d-1}(j \leftarrow i, v \leftarrow u)$ and $\mathcal{W}_{d-1}(j' \leftarrow i, v' \leftarrow u)$.



Neighborhood tree matching algorithm, main result

NTMA algorithm: $\mathcal{S} = \varnothing$. For all pairs $(i, u) \in V(G_1) \times V(G_2)$ whose $d-$neighborhoods $\mathcal{N}_i$ and $\mathcal{N}_u$ are trees:
If there exists $j \neq j' \overset{G_1}{\sim} i$, $v \neq v' \overset{G_2}{\sim} u$ such that $\mathcal{W}_{d-1}(j \leftarrow i, v \leftarrow u) > \tau$ and $\mathcal{W}_{d-1}(j' \leftarrow i, v' \leftarrow u) > \tau$, then add (i, u) to $\mathcal{S}$.

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Theorem

Assume $\lambda s > 1$, $\lambda \in (1, \lambda_0]$ and $s \in (s^*(\lambda), 1]$. Then for $d = \Theta(\log n)$ and $\tau = \Theta(\gamma^{d-1})$, with high probability:

$$\frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{(i,\sigma(i)) \in \mathcal{S}} = \Omega(1) \quad \textit{and} \quad \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\exists u \neq \sigma(i), (i,u) \in \mathcal{S}} = o(1).$$

In practice...

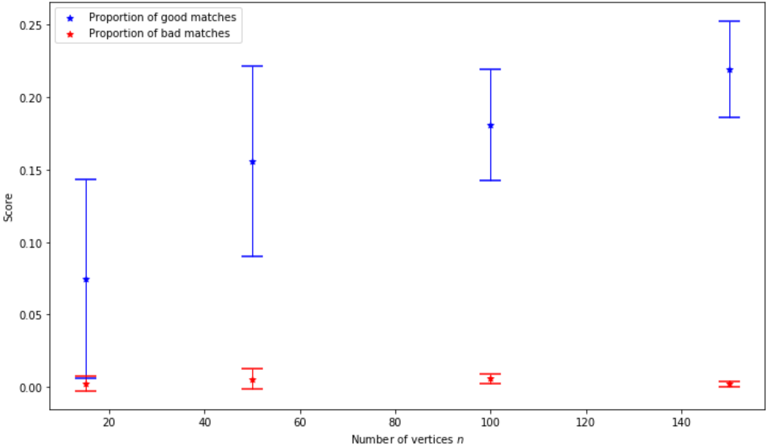

Figure: Mean score of NTMA-2 for $\lambda = 2.1$, $d = 5$, and $s = 1$ (isomorphism case).

In practice...

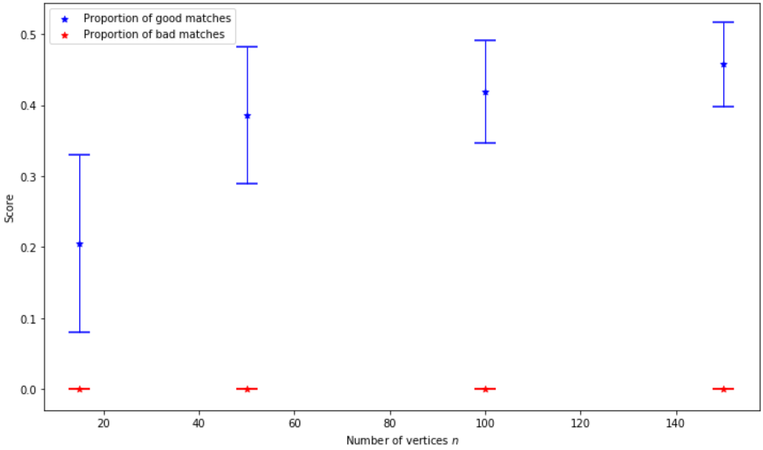

Figure: Mean score of NTMA-2 for $\lambda = 2.1$, $d = 5$, and $s = 0.95$.

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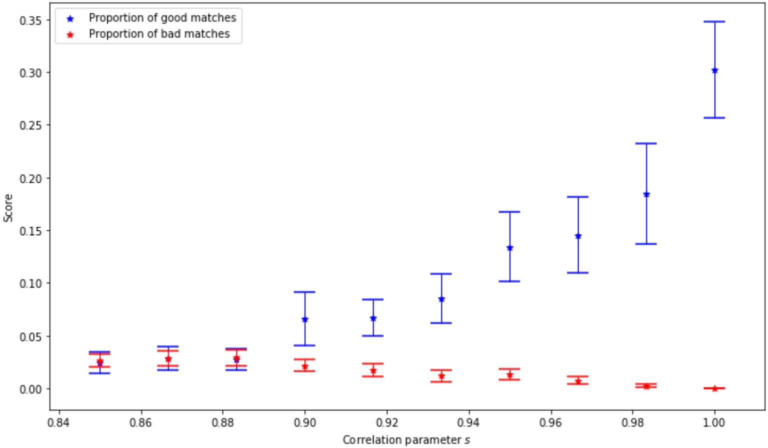

Figure: Mean score of NTMA-2 for $n = 150$, $\lambda = 1.4$, $d = 5$, and varying s.

Take-home messages

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- Outlook: boundaries of phases in (λ, s) diagram, in particular IT-feasibility and poly-time feasibility of partial alignment, and correlation detection in trees.

Thank you!